Optimal Hotelling Auctions

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LSE Reading Group

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Motivation

Sellers often have a quality line of products to offer.

- Some products may be horizontally differentiated, whilst others are homogeneous.
- Examples: Radio spectra, search words, cloud computing capacity...
- But what is the optimal method of selling such goods?

Optimal multi-product selling mechanisms are hard and not well understood.

- Optimal mechanisms may include randomisation.
- Counterintuitive results, increasing buyer valuations may *decrease* seller revenue (Hart & Reny, 2015).
- Monotonicity does not tractably extend and cannot restrict to local downward ICs even under regularity assumption.

The Paper: A Roadmap

Two horizontally differentiated goods, several of each 'type'. Collapse multivariate valuations onto Hotelling line. Seller has goods at extreme ends of line. How to deal with difficulties of multiple products?

- **Thm 1**: Characterise optimal mechanism by novel saddle-point property based on worst-off type.
- **Lem 4**: Deal with monotonicity condition problems.
- Using tractable monotonicity condition, iron out virtual valuations and maximise pointwise.

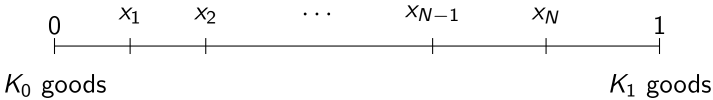
Find the optimal mechanism...

- **Prop 2**: Two independent auctions at extremes of Hotelling line optimal for seller iff no buyer has positive valuation for both goods.
- **Thm 2/Prop 3**: Defines optimal mechanism when indep auctions non-optimal: randomises allocations.

Thm 3: Implementation via a two-stage clock auction.

Setup: Environment

Two types of good at extremes of Hotelling line, indexed by $\ell \in \{0, 1\}$ with total supply K_0 and K_1, respectively. There are N buyers, 1 seller. Assume for presentation that $K_0 = K_1 = N$ (monopoly problem).



- Buyers' commonly value both goods at v. Unit demand for any good.
- Location $x_i \in [0, 1]$ private info of buyers. Common prior $x_i \overset{i.i.d}{\sim} F$, with full-support density f.
- Assume F is **Myerson Regular —** any non-monotonicty or randomised allocation inherent to *countervailing incentives*...
- For type x, probability of getting good $\ell \in \{0, 1\}$ is q_ℓ. If transfer t to seller, expected payoff

$$q_0(v - x) + q_1(v - [1 - x]) - t$$

Setup: Mechanisms

Set of outcomes $\{(0,0),(1,0),(0,1)\}$. Outcome (a, b) represents getting a units of good 0 and b units of good 1.

- A direct symmetric mechanism is $\langle \mathbf{Q}, T \rangle$,

$$\mathbf{Q} = (Q_0, Q_1) : [0,1]^N \to \Delta(\{(0,0),(1,0),(0,1)\})$$
$$T : [0,1]^N \to \mathbb{R}$$

- Define interim allocation and transfer (q_0, q_1, t) and utility u as

$$\begin{aligned} q_0(x_n) &= \mathbb{E}_{\mathbf{x}_{-n}}[Q_0(x_n, \mathbf{x}_{-n})] \\ q_1(x_n) &= \mathbb{E}_{\mathbf{x}_{-n}}[Q_1(x_n, \mathbf{x}_{-n})] \\ t(x_n) &= \mathbb{E}_{\mathbf{x}_{-n}}[T(x_n, \mathbf{x}_{-n})] \\ u(x, x') &= q_0(x')[v - x] + q_1(x')[v - (1 - x)] - t(x') \end{aligned}$$

- Let $u(x) = u(x, x)$ be truthful utility. IC and IR if for all $x, x' \in [0,1]$, $u(x) \geq u(x, x')$ and $u(x) \geq 0$.

Implementability

Lemma 1

A direct mechanism $\langle \mathbf{Q}, T \rangle$ is incentive compatible if and only if for all $x, x' \in [0, 1]$,

$$q_1(x) - q_0(x) \text{ is increasing} \qquad (CM)$$

$$u(x) = u(x') + \int_{x'}^{x} [q_1(y) - q_0(y)] dy \qquad (Env)$$

- Fix **critical type** $x' \in [0, 1]$ arbitrarily for now.
- **Problem 1:** (CM) unwieldy: ideally $q_1(x)$ and $-q_0(x)$ non-decreasing in x, but not clear from constraint.
- **Problem 2:** What should the critical type be? Easy to satisfy IR if pick worst-off type, but who is this now? Critical worst-off type endogenous to allocation rule by (CM)!
- **Problem 3:** Have to think about more than just local downward IC constraints (see next slide).

Countervailing Incentives

By (Env), in any IC mech $\langle \mathbf{Q}, T\rangle$, seller revenue is

$$\begin{aligned} R(\mathbf{Q}, T; x') &= N\left(\int_0^1 [q_0(x)\Psi_0(x,x') + q_1(x)\Psi_1(x,x')]dF - u(x')\right) \\ &= N(\tilde{R}(\mathbf{Q}; x') - u(x')) \end{aligned}$$

where

$$\Psi_0(x,x') = \begin{cases} v - \psi_B(x) & \text{if } x > x' \\ v - \psi_S(x) & \text{if } x \leq x' \end{cases}$$

$$\Psi_1(x,x') = \begin{cases} v - (1-\psi_B(x)) & \text{if } x > x' \\ v - (1-\psi_S(x)) & \text{if } x \leq x' \end{cases}$$

- Downward IC for $x > x'$, upward IC for $x \leq x'$.
- No natural way to restrict to only one direction $\implies$ countervailing incentives inherent (discontinuous at x').

Critical Candidate: Worst-Off Types

Worst-Off Types

For IC mech $\langle \mathbf{Q}, T \rangle$, define set of worst-off types $\Omega(\mathbf{Q}) = \arg\min_{x \in [0,1]}\{u(x)\}$. Then,

- **Lemma 2:** $\Omega(\mathbf{Q})$ depends on $\mathbf{Q}$.
- **Lemma 3:** $\Omega(\mathbf{Q}) = \arg\min_{x' \in [0,1]} \tilde{R}(\mathbf{Q}, x')$

Seller's revenue is $R(\mathbf{Q}, T) = N(\tilde{R}(\mathbf{Q}, x') - u(x'))$. For optimal mech $\langle \mathbf{Q}^*, T^* \rangle$, $u(\omega) = 0$ for all $\omega \in \Omega(\mathbf{Q}^*)$. Seller solves

$$\max_{\mathbf{Q}} \tilde{R}(\mathbf{Q}, \omega), \qquad \text{st. } \omega \in \Omega(\mathbf{Q})$$

Lemma 3 implies

$$\mathbf{Q}^* = \arg\max_{\mathbf{Q}} \min_{x' \in [0,1]} \tilde{R}(\mathbf{Q}, x')$$

Saddle-Point Theorem

Seller revenue maximised by $\mathbf{Q}^* = \arg\max_{\mathbf{Q}} \min_{x' \in [0,1]} \tilde{R}(\mathbf{Q}, x')$ $(\dagger)$
Divide into a saddle-point problem.

Definition

A *Saddle-Point* of the seller's virtual surplus $\tilde{R}(\mathbf{Q}, x')$ is an allocation rule and critical type $(\mathbf{Q}^*, \omega^*)$ such that

$$\mathbf{Q}^* = \arg\max_{\mathbf{Q}} \tilde{R}(\mathbf{Q}, \omega^*)$$

$$\omega^* = \arg\min_{x'} \tilde{R}(\mathbf{Q}^*, x')$$

Theorem 1

The allocation rule of any saddle-point satisfies $(\dagger)$. Conversely, if $\mathbf{Q}^*$ solves $(\dagger)$, there exists a critical type, ω^*, such that $(\mathbf{Q}^*, \omega^*)$ is a saddle point. A solution to $(\dagger)$ exists, and so there exists a saddle point.

Strong Monotonicty

Virtual surplus function is

$$\tilde{R}(\mathbf{Q}, x') = \int_0^1 [q_0(x)\Psi_0(x, x') + q_1(x)\Psi_1(x, x')]dF$$

Pointwise maximisation liable to violate monotonicity $\implies$ problem is inherently non-regular.

- Want to iron out objective, but monotonicity constraint $q_1(x) - q_0(x)$ stops us ironing Ψ_0 and Ψ_1 separately.

Lemma 4

Without loss of generality, we can restrict attention to allocation rules $\mathbf{Q}$ such that $q_1(x)$ and $-q_0(x)$ are weakly increasing (strong monotonicity).

Ironing

For allocations such that $q_1(x)$ and $-q_0(x)$ are monotone increasing, can iron $\Psi_0(x,x'), \Psi_1(x,x')$ separately.

Ironing

For any critical type $x' \in (0,1)$, there exists an interval $I(x') = [\underline{x}(x'), \overline{x}(x)] \ni x'$ such that for $\ell \in \{0,1\}$ the ironed virtual types are,

$$\overline{\Psi}_\ell(x,x') = \Psi_\ell(x,x') + \mathbb{1}_{\{x \in I(x')\}}(z_\ell(x') - \Psi_\ell(x,x'))$$

where $z_0(x'), (z_1(x'))$ are continuous and increasing (decreasing) in x'. The endpoints $\underline{x}(x'), \overline{x}(x')$ are continuous and increasing in x'.

Parameters

Ironing params satisfy $z_0(x') + z_1(x') = 2v - 1$ for any x'. Additionally, as cts., by IVT there exists $\hat{x}_A$ such that $z_0(\hat{x}_A) = z_1(\hat{x}_A)$. Such $\hat{x}_A$ corresponds to ironing interval $I(\hat{x}_A) \subset (0,1)$. Details

(Non)-Optimality of Independent Auctions

Propositon 2

Running two independent Myerson optimal auctions for the goods $\ell \in \{0, 1\}$ is optimal if and only if $v \leq \frac{1}{2}$. If this holds, the seller never serves a non-null set of buyers.

- By **Thm 1**, indep auctions optimal $\iff$ never serve some mass of buyers.
- Never serve some mass of buyers $\iff$ $z_0(x'), z_1(x') \leq 0$ and $I(x') \subset (0, 1)$. Pick $x' = \hat{x}_A$ such that $z_0(\hat{x}_A) = z_1(\hat{x}_A)$ so $I(\hat{x}_A) \subset (0, 1)$.
- As $z_0 + z_1 = 2v - 1 \implies z_0(\hat{x}_A) = z_1(\hat{x}_A) = v - \frac{1}{2}$, $z_0(\hat{x}_A), z_1(\hat{x}_A) \leq 0$ if and only if $v \leq \frac{1}{2}$

Remark

The condition $v \leq \frac{1}{2}$ holds if and only if two independent auctions are efficient (with reserve price 0)

Lottery-Augmented Auctions

By Proposition 2, if $v > \frac{1}{2}$, indep auctions non-optimal.

Proposition 3/Theorem 2

If $v > \frac{1}{2}$ and $K_0 = K_1 = N$ the optimal mechanism (q_0^*, q_1^*, t^*)

$$q_0^*(x) = \begin{cases} 1 & \text{if } x < \alpha \\ \frac{1}{2} & \text{if } x \in [\alpha, \beta] \\ 0 & \text{if } x > \beta \end{cases}, \quad t^*(x) = \begin{cases} v - \psi_S^{-1}\left(\frac{1}{2}\right) & \text{if } x < \alpha \\ v - \frac{1}{2} & \text{if } x \in [\alpha, \beta] \\ v - \left[1 - \psi_B^{-1}\left(\frac{1}{2}\right)\right] & \text{if } x > \beta \end{cases}$$

and $q_1^*(x) = 1 - q_0^*(x)$, for some $0 < \alpha < \beta < 1$.

- Proof computes pointwise maximimising ex-post allocation for every critical type. Then looks for allocation rule and critical type satisfying saddle-point condition.

Illustration and Three-Price Implementation

Suppose $F \sim U[0,1]$ and $v = 1$. Then randomised allocations in interval $\left[\frac{1}{4}, \frac{3}{4}\right]$.

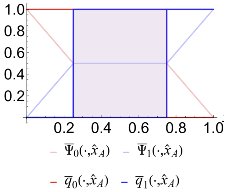


Figure 1

- Prices $p_0 = p_1 = v - \frac{1}{4}$ for the pure goods, $0, 1$.
- Price $p_L = v - \frac{1}{2}$ for lottery good.
- Can generally decompose into three prices whenever $K_0 = K_1 = N$ and $v > \frac{1}{2}$.
- Without this condition, more complex implementation needed...

Two-Stage Clock Implementation

Argue in paper that clock auctions have advantages including privacy of winner.

- Suppose $v > \frac{1}{2}$ and $\min\{K_0, K_1\} < N$ (so no two or three-price implementation), can we preserve privacy of winners?

Two-Stage Clock Auction

- **First Stage** In the *coarse bidding* stage, agents either ask for good 0 or 1, or neither. If they don't ask for either good, they are allocated the goods with some probability based on demand for goods 1 and 0.
- **Second stage** If supply can meet demand of good $\ell \in \{0, 1\}$ based on coarse bids, good is immediately allocated to agents. Otherwise, agents are entered into an ascending price auction with a reserve price.

Theorem 3

The equilibrium of the two-stage clock auction implements the optimal mechanism in weakly dominant strategies.

Appendix A: The Single-Good Problem

Suppose N buyers for a single indivisible good. Privately known valuations $x \sim F$, with density f such that $\text{supp}(f) = [\underline{x}, \overline{x}]$. By revel'n principle, search for feasible/implementable direct mechanism $\langle Q, T \rangle$.

- **Implementability:** Let (q, t) be interim allocation and transfers for buyer and u be interim payoff from truthful reporting (same for all i), then $\langle Q, T \rangle$ is IC+IR iff,

$$\forall x < x', \qquad q(x) \leq q(x') \qquad (M)$$

$$\forall x, x', \qquad t(x) = q(x)x - u_i(x') - \int_{x'}^{x} q_i(y)dy \qquad (Env)$$

$$\forall x, \qquad u_i(x) \geq 0 \qquad (IR)$$

- Refer to x' as the **critical type** (show how to choose later).

Appendix A: The Single-Good Problem

For direct mech $\langle Q, T\rangle$ and any critical type x', seller revenue is (by (ENV)),

$$R(Q,T;x') = N\int_{[\underline{x},\overline{x}]} t(x)dF = N\left(\underbrace{\int_{[\underline{x},\overline{x}]}[\Psi(x,x')q(x)]dF}_{:=\tilde{R}(Q,x')} - u(x')\right)$$

where

$$\Psi(x,x') = \begin{cases} x - \frac{1-F(x)}{f(x)} & \text{if } x > x' \\ x + \frac{F(x)}{f(x)} & \text{if } x \leq x' \end{cases} = \begin{cases} \psi_B(x) & \text{if } x > x' \\ \psi_S(x) & \text{if } x \leq x' \end{cases}$$

- By (M), $q(x)$ is increasing. As $\psi_S(x) > \psi_B(x)$, for fixed rule Q, $\tilde{R}(Q,x')$ is minimised at $x' = 0$, the worst-off type!
- $\implies$ standard problem is to solve $Q^* = \arg\max_Q \min_{x'} \tilde{R}(Q,x')$. Solvable by ironing out virtual surpluses and maximising pointwise.

Appendix B: Ironing

In paper, show critical worst-off type lies in some interval $[\hat{x}_0, \hat{x}_1] \ni \hat{x}_A$.

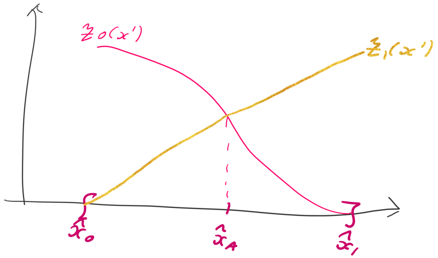


Figure 2

- At $\hat{x}_0$, $z_1(\hat{x}_0) = 0, z_0(\hat{x}_0) > 0$.
- If $x' \in (\hat{x}_0, \hat{x}_A)$, $z_0(x') > z_1(x') > 0$.
- At $\hat{x}_A$, $z_0(\hat{x}_A) = z_1(\hat{x}_A) > 0$.
- If $x' \in (\hat{x}_0, \hat{x}_A)$, $z_1(x') > z_0(x') > 0$.
- At $\hat{x}_1$, $z_0(\hat{x}_0) = 0, z_1(\hat{x}_0) > 0$.

Appendix B: Ironing

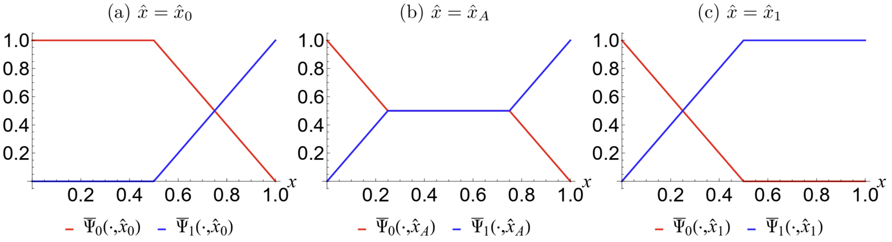


Figure 3: Uniform locations, $v = 1$.

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