A Characterization for Optimal Bundling of Products with Nonadditive Values

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Introduction

- In this paper, Ghili seeks to elicit the conditions under which pure bundling, i.e. only selling the package of all available products in one bundle, is optimal.
- He shows that under monotonic preferences and single-peaked profits, pure bundling is optimal if and only if it leads to higher sale volume.

Set Up

- We have a monopolist with n products, that can form bundles. The set of all bundles is $\mathcal{B} = \{b | b \subseteq \{1, \ldots, n\}\}$, the grand bundle is $\bar{b}$.
- It sells to a mass of customers, each with a type $t \in T$, distributed according to probability distribution $f(\cdot)$.
- Each type t values bundle b as $v(b, t)$. For disjoint bundles b and b', a type already possessing bundle b' values bundle b as follows $v(b, t | b') = v(b \cup b', t) - v(b', t)$.
- Producing each product costs c_i and the cost of producing bundle b is $\sum_{i \in b} c_i$.

Set up

- A direct selling mechanism $(a(\cdot), p(\cdot))$ is defined by:
  - An allocation rule: $a(\cdot) : T \to \Delta(\mathcal{B})$.
  - A pricing rule: $p(\cdot) : T \to \mathbb{R}$.
- A mechanism is individually rational (IR) if $\forall t \in T : v(a(t), t) - p(t) \geq 0$.
- A mechanism is incentive compatible (IC) if $\forall t, t' \in T : v(a(t), t) - p(t) \geq v(a(t'), t) - p(t')$.
- The monopolist wants to find the optimal mechanism (a^*, p^*) that solves:

$$max_{a,p} \int_t [p(t) - c_{a(t)}] f(t) dt \quad \text{subject to IC and IR.}$$

Main assumptions and main Theorem

- *Assumption 1*: Monotonicity of $v(b, t)$ in t.
- *Assumption 2*: Quasi-concavity of profits in p.
- **Theorem**: Suppose that assumptions 1 and 2 hold. Then, for an optimal mechanism (a^*, p^*) to involve pure bundling, it is necessary and sufficient that

$$D^*(\bar{b}) \geq max_{b \in \mathcal{B}} D^*(b)$$

.

Preliminary lemmas and definitions

- The virtual value function ϕ is defined as $\phi(b,t) = v(b,t) - \frac{1-F(t)}{f(t)} v_t(b,t) - c_b$.
- *Lemma*: For any IC mechanism (a,p), we have $\mathbb{E}[p(t) - c_{a(t)}] = \mathbb{E}[\phi(a,t)] - [v(a(\underline{t}),\underline{t}) - p(\underline{t})]$.

Preliminary lemmas and definitions

- *Lemma*: Consider $\phi(b, \cdot)$ as a function of t. As long as $b \neq \emptyset$, monotonicity and quasi-concavity imply that $\phi(b, \cdot)$ crosses zero only once from below. The same is true of any virtual value function $\phi(b, \cdot | b^c)$ for all nonempty b.

Preliminary lemmas and definitions

- *Lemma*: Consider $\phi(b, \cdot)$ as a function of t. As long as $b \neq \emptyset$, monotonicity and quasi-concavity imply that $\phi(b, \cdot)$ crosses zero only once from below. The same is true of any virtual value function $\phi(b, \cdot | b^c)$ for all nonempty b.
  - *Proof*: Rewrite the virtual value function as $\phi(b, t) = v(b, t) - \frac{1 - F_b(v(b,t))}{f_b(v(b,t))} - c_b$. It holds that $\{t \in T : \phi(b, t) = 0\} = \{t \in T : \frac{\partial \pi_b(t)}{\partial t} = 0\}$. By quasi-concavity, $\{t \in T : \frac{\partial \pi_b(t)}{\partial t} = 0\}$ only has one element, and thus $\phi(b, t) = 0$ occurs for only one t, denoted $t^*(b)$.

Proof of Sufficiency - Intuitively

- To prove sufficiency, first Ghili demonstrates that there is an upper bound on the value that the monopolist can extract from each consumer, and that this upper bound corresponds to the value it extracts when selling the pure bundle.
- Since this upper bound is achievable, Ghili demonstrates that the monopolist has an interest in offering the pure bundle.

Proof of Sufficiency

- First, demonstration that $\forall b, t, \quad \phi(b,t) \leq max\{0, \phi(\bar{b}, t)\}$.
  - $D^*(\bar{b}) \geq D^*(b)$ implies that $t^*(\bar{b}) \leq t^*(b)$. Since $\pi_{\bar{b}}(t) \equiv \pi_b(t) + \pi_{b^c}(t|b)$, we have $t^*(b^c|b) \leq t^*(\bar{b}) \leq t^*(b)$.
  - From the definition of $\phi(b,t)$ and $\phi(b^c, t|b)$, we have $\phi(\bar{b}, t) = \phi(b,t) + \phi(b^c, t|b)$.
  - When $t \leq t^*(b)$, by definition of $t^*(b)$, $\phi(b,t) \leq 0 \leq \max\{0, \phi(\bar{b}, t)\}$.
  - When $t > t^*(b) \geq t^*(b^c|b)$, $\phi(b^c, t|t) > 0$ meaning that $\phi(b,t) < \phi(\bar{b}, t) \leq \max\{0, \phi(\bar{b}, t)\}$.
- This allows us to find $\mathbb{E}[p^*(t) - c_{a^*(t)}] \leq \mathbb{E}[\phi(a^*(t), t)] \leq \mathbb{E}[\max\{0, \phi(\bar{b}, t)\}]$. Thus, the pure bundling is an upper bound on the monopolist's expected profits.
- Since this mechanism is feasible by setting $a(t) = \emptyset$ if $t < t^*(\bar{b})$ and $a(t) = \bar{b}$ otherwise, we get the proof of sufficiency.

Proof of Necessity

- Demonstration that if there exists $b \neq \bar{b}$ such that $D^*(b) > D^*(\bar{b})$, then pure bundling is strictly suboptimal.
- Since $D^*(b) > D^*(\bar{b})$ yields $t^*(b) < t^*(\bar{b})$, we implement the allocation $a(t) = b$ if $t^*(b) \leq t < t^*(\bar{b})$ and $a(t) = \bar{b}$ otherwise.
- Setting $\bar{a}(\cdot)$ the allocation such that $\bar{a}(t) = \emptyset$ if $t < t^*(\bar{b})$ and $\bar{a}(t) = \bar{b}$ otherwise, we observe that $v(\bar{a}(\underline{t}), \underline{t}) - p(\underline{t}) = v(a(\underline{t}), \underline{t}) - p(\underline{t}) = 0$.
- For $t \in [t^*(b), t^*(\bar{b}))$, $\phi(a, t) \geq 0 > \phi(\bar{a}, t)$. For $t \notin [t^*(b), t^*(\bar{b}))$, $\phi(a, t) = \phi(\bar{a}, t)$.
- Therefore, $\mathbb{E}[\phi(a, t)] > \mathbb{E}[\phi(\bar{a}, t)]$.