Selected Facts

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Micro Theory Reading Group
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Introduction

- Persuasion model with sender and receiver
  * Sender (S) privately informed about state of the world
  * S selects pieces of hard evidence to show to Receiver (R) and recommends action
  * Based on evidence R takes binary action "accept" a / "reject" r
- S and R have contrasting preferenes
  * S biased towards accepting: in some states S prefers a while R prefers r
- Paper contributions are characterisations of
  * necessary and sufficient conditions for 'subversion'
  * S-optimal strategies
  * R-optimal ex-ante *restriction* of admissible facts

Outline of Talk

Introduction

Model

Results

Information and Preferences

- S is knows state $x \in X$, recommends $d \in \{a, r\}$
- R has full-support prior $p(x)$
- Given x, payoffs depend on *implemented* d
  * If $d = r$, both get zero
  * If $d = a$ S gets $v(x)$, R gets $u(x)$
- Agreement subsets of X are

$$A = \{x \in X : u(x) > 0 \ \& \ v(x) > 0\}$$
$$R = \{x \in X : u(x) < 0 \ \& \ v(x) < 0\}$$

- Contrast area is $C = X \setminus A \cup R$

S's Strategy and Timing

- Given x, S can produce a report

$$m \in \mathcal{M}(x) = \underbrace{\mathcal{F}(x)}_{\text{available facts}} \times \underbrace{\{a, r\}}_{\text{recommendation}} \times \underbrace{\boldsymbol{M}}_{\text{cheap-talk message}}$$

- Distiction between *hard* and *soft* evidence
  - * Reporting ϕ means $p(x|\phi) = 0$ whenever $\phi \notin \mathcal{F}(x)$
  - * Soft-information to handle mixing
- Before learning x, S commits to reporting strategy

$$\sigma = \{\sigma(x)\}_{x \in X}, \text{ where } \sigma(x) \in \Delta(\mathcal{M}(x))$$

- S's problem is Cartesian if
  1. $X = \prod_{i=1}^{n} X_i$, and each x_i is *decision relevant* aspect
  2. $\mathcal{F}(x) = \{\phi \subseteq \{x_1, \dots x_n\} : |\phi| = k\}$ for some $k \geq 1$

R's Strategy

- R forms posterior $p(x|\sigma)$, optimally selects a or r
- Given realised report m, R accepts iff

$$\mathbb{E}[u(x)|m : d(m) = a] \geq 0 \geq \mathbb{E}[u(x)|m : d(m) = r] \qquad (1)$$

- A strategy σ is subversive if R accepts whenever $x \in A \cup C$ and rejects if $x \in R$

Graph-Theoretic Setup

- Consider bipartite graph G with $V(G) = A \cup C$
- Nodes connected iff states are "fact-poolable":

$$E(G) = \{\{x, x'\} : x \in C, x' \in A, \mathcal{F}(x) \cap \mathcal{F}(x') \neq \varnothing\}$$

- Neighbours of $S \subseteq C$ are

$$N(S) = \{x \in V(G) : \{x, x'\} \in E, x' \in S\} \subseteq A$$

Graph-Theoretic Preliminaries

– Matching is $M \subseteq E$ such that

$$\{x, x'\} \in M \implies \{x, x''\} \notin M \;\&\; \{x', x''\} \notin M, \forall x'' \in V(G)$$

– Full-domain injection $f : C \mapsto A$ is "C-perfect" M

Theorem (Hall's Marriage Theorem)

Given bipartite $G = (A, C)$ a C-perfect matching exists iff

$$|N(S)| \geq |S| \quad \textit{for all} \quad S \subseteq C. \tag{HC}$$

Proof.

$\implies$: $|S| > |N(S)|$ means $f(x) = \varnothing$ for some $x \in S$

$\impliedby$: by induction on $|C|$ (not discussed, check here) ■

Outline of Talk

Introduction

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Subversion Iff Pooling Favours a

Theorem (Subversive Strategy)

A subversive reporting strategy exists iff

$$\mathbb{E}[u|S \cup N(S)] \geq 0 \quad \textit{for all non empty} \quad S \subseteq C \qquad (2)$$

Outline of proof before details:

– Sufficiency:

Step 1: Define auxiliary graph ${}_cG$, suppose (HC) holds, find subversive σ

Step 2: Show (HC) holds on ${}_cG$ if (2) holds

– Necessity:

Step 1: Argue subversive σ recommends a for $x \in C, x' \in N(\{x\})$

Step 2: Use LIE

Proof: Auxiliary Graph

– Define $w(x) = |p(x)u(x)| \in \mathbb{Q}$ (wlog)

– Pick largest $h \in \mathbb{Q}$ so that $w(x)/h = n(x) \in \mathbb{N}$

Define auxiliary graph:

– Nodes are clones of all $x \in A \cup C$ of the form $\{{}^i_c x\}_{i=1}^{n(x)}$

– Cloned graph is still bipartite

$$ {}_cA = \bigcup_{x \in A} \{{}^i_c x\}_{i=1}^{n(x)} \text{ and } {}_cC = \bigcup_{x \in C} \{{}^i_c x\}_{i=1}^{n(x)} $$

– Preserve original edges:

$$ \{{}^i_c x, {}^j_c x'\} \in {}_cE, \forall i, j \in \{1, \ldots, n\} \text{ iff } \{x, x'\} \in E $$

Proof: If (1/2)

- Suppose C-perfect ${}_cM$ holds on ${}_cG$
- For any matched $\{x, x'\}$ report m as follows:
  * Fact $\phi \in \mathcal{F}(x) \cap \mathcal{F}(x')$
  * Soft-info pins down match with $\mu : {}_cE \mapsto \boldsymbol{M}$
  * Recommendation $d(m) = a$
- Set $\sigma(y)[m] = {}^{h}/_{w(y)}$ so that posterior is

$$\Pr[A|d(m) = a, \phi] = \frac{\frac{h}{w(x')}p(x')}{\frac{h}{w(x)}p(x) + \frac{h}{w(x')}p(x')} = -\frac{u(x)}{u(x') - u(x)}$$

- So $\mathbb{E}[u|d(m) = a, \phi] = 0$ and (1) holds: σ subversive

Proof: If (2/2)

- Pick ${}_cS \subseteq {}_cC$, let $S = \{x \in C : \exists i : {}^i_c x \in {}_cS\}$ and
$$ {}_cS' = \bigcup_{x \in S} \{{}^i_c x\}_{i=1}^{n(x)} $$
- All clones of x have same neighbours so
$$ \begin{aligned} |N({}_cS)| &= |N({}_cS')| \\ &= \sum_{x' \in N(S)} \frac{w(x')}{h} \geq_{(2)} \sum_{x \in S} \frac{w(x)}{h} \\ &= |{}_cS'| \geq |{}_cS| \end{aligned} $$
- So (HC) holds on ${}_cG$

Proof: Only If

– $x \in C$ means for some $x' \in N(\{x\})$

$$\mathrm{supp}_a(\sigma(x)) \cap \mathrm{supp}_a(\sigma(x')) \neq \varnothing$$

– $\pi(x)$ set of all such x'. Partition

$$S \cup N(S) = (S \cup \pi(S)) \cup (N(S) \setminus \pi(S))$$

– $u > 0$ on $N(S) \setminus \pi(S) \subseteq A$

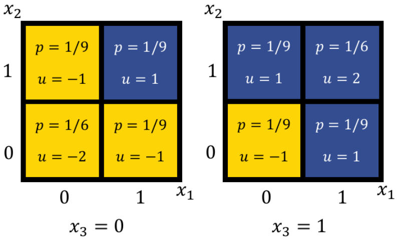
– Term in EU with ambiguous sign is

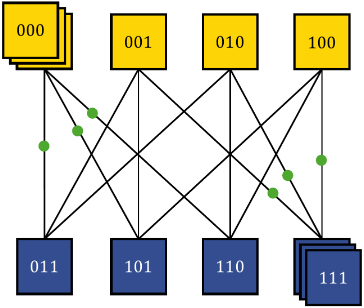
$$\mathbb{E}[u|S \cup \pi(S)] = \sum_{m \in \mathrm{supp}_a \sigma(S \cup \pi(S))} \mathbb{P}[m|S \cup \pi(S)]\, \mathbb{E}[u|m] \geq 0$$

– Non-negativity follows from σ subversive

Example of Subversion

$$X = \{0,1\}^3 \text{ and } \mathcal{F}(x) = \{x_1, x_2, x_3\}$$





$$\sigma(x = (0,0,0)) = \begin{cases} \phi = x_1, \hat{\mu} \in \{\mu(\{{}^i_c x, (0,1,1)\})\}_{i=1}^3, \mathbb{P}(\hat{\mu}) = {}^1\!/_9 \\ \phi = x_2, \hat{\mu} \in \{\mu(\{{}^i_c x, (1,0,1)\})\}_{i=1}^3, \mathbb{P}(\hat{\mu}) = {}^1\!/_9 \\ \phi = x_3, \hat{\mu} \in \{\mu(\{{}^i_c x, (1,1,0)\})\}_{i=1}^3, \mathbb{P}(\hat{\mu}) = {}^1\!/_9 \end{cases}$$

Optimal Strategy is Maximal-Weight Matching

Theorem (Optimal Strategy Representation)

S's optimal strategy is a maximal-weight matching on ${}_{\eta}G$ *and a maximal-cardinality matching on* ${}_{\eta}G$.

- Kőnig-Ore formula yields # of unmatched vertices given ${}_{c}M$ with maximal cardinality

$$\mathrm{def}({}_{c}G) = \max_{S \subseteq {}_{c}C} [|S| - |N(S)|]$$

- Depends on primitive G, not on v
- $\mathbb{E}[u|\text{optimal } \sigma] = \mathbb{E}[u|x \in A \cup C] + \mathrm{def}({}_{c}G)$
- σ subversive iff $\mathrm{def}({}_{c}G) = 0$

R-Optimal Fact Restriction is Hall-Deficit Minimiser

Theorem (Design of Admissible Facts)

Given problem $\mathcal{P}$, the optimal set of admissible facts $\mathcal{F}^*(\mathcal{P})$ solves

$$\begin{aligned} \mathcal{F}^*(\mathcal{P}) = \arg\max_{\mathcal{F}' \subseteq \mathcal{F}} \delta_H(\mathcal{F}') \\ s.t. \quad \mathcal{F}'(x) \neq \varnothing \quad \textit{for all} \quad x \in X. \end{aligned} \tag{3}$$

$\mathcal{F}^*(\mathcal{P})$ is invariant to cardinal transformations of v.

- Special case: when $k = 1$, reduce reporting to unique fact with largest δ_H^i
- "Least-poolable" fact