# Nash Bargaining Solution: Threats and Counter-Threats

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In class, we saw that given a bargaining problem $(X, u^1, u^2, d^1, d^2)$, an allocation $x^* \in X$ was a Nash bargaining solution if and only if for each $i \in \{1, 2\}$, for any $\tilde{x} \in X$ and $p \in [0, 1]$,

$$pu^i(\tilde{x}) + (1-p)u^i(d) > u^i(x^*) \implies pu^{-i}(x^*) + (1-p)u^{-i}(d) > u^{-i}(\tilde{x})$$

The representation in terms of threats and counterthreats comes from Rubinstein, Safra, and Thomson (1992)[1]. Several students asked why it must be the case that the counterthreat's probability of disagreement is necessarily the same as the probability of disagreement in the threat. That is, why must the probability $p$ be used for both $i$ and $-i$.

The reason for this is that in the axiomatic definition of Nash bargaining, we implicitly impose the two agents wield the same bargaining power through the symmetry axiom. The symmetry axiom imposes that if for any $x \in X$ there exists $\hat{x} \in X$ such that $(u_1(x), u_2(x)) = (u_2(\hat{x}), u_1(\hat{x}))$ and if $d_1 = d_2$ then the NBS, $x^* \in X$, satisfies $u_1(x^*) - u_1(d_1) = u_2(x^*) - u_2(d_2)$. That is, if the agents have the same utility possibility sets and the same disagreement point, then they should get the same utility benefit from bargaining. Whilst this intuitively captures the idea of equal bargaining power, we formally show the equivalence in the following exercise.

**Exercise:** Consider the following solution concept for a given bargaining problem $(X, u^1, u^2, d^1, d^2)$. Say that $x^* \in X$ is an $\alpha$-solution if

$$x^* \in \arg\max_{x \in X} (u_1(x) - u_1(d_1))^{\alpha}(u_2(x) - u_2(d_2))^{1-\alpha}$$

For some $\alpha \in (0, 1)$. Often in applied work, the above solution concept is used to model different bargaining power between the two agents. Specifically, $\alpha$ represents the *bargaining weight* on agent 1 and $1 - \alpha$ represents the weight on agent 2.

1. Show that the $\alpha$-solution concept satisfies the symmetry axiom if and only if $\alpha = 1/2$.

2. Show that the $\alpha$-solution concept may be represented in terms of threats and counterthreats where the probability of disagreement for player 1's (counter)threats, $1 - p_1$, and player 2's probability of disagreement, $1 - p_2$, satisfy $p_1 = p_2^{\frac{1-\alpha}{\alpha}}$.

[1] Rubinstein, A., Safra, Z., & Thomson, W. (1992). On the Interpretation of the Nash Bargaining Solution and Its Extension to Non-Expected Utility Preferences. *Econometrica, 60*(5), 1171–1186.

3. Notice, $\alpha = 1/2$ if and only if $p_1 = p_2$ so that $p_1 = p_2$ if and only if the $\alpha$-solution is symmetric.

**Proof:** For brevity, set $\Delta_i(x) := u_i(x) - u_i(d_i) \geq 0$.

**1) Symmetry axiom $\iff \alpha = \frac{1}{2}$.** *($\Rightarrow$)* We prove the contrapositive. That is, we show $\alpha \neq 1/2 \implies$ Not Symmetric.

Consider a symmetric problem: $d_1 = d_2 =: d$ and the feasible utility set symmetric around the diagonal. We consider a subset of bargaining problems. Specifically, consider the set of bargaining problems which have symmetric frontier $\Delta_1 + \Delta_2 = C$ with $C > 0$. The $\alpha$-program becomes

$$\max_{0<\Delta_1<C} f(\Delta_1) := \Delta_1^{\alpha}(C - \Delta_1)^{1-\alpha}.$$

Compute

$$f'(\Delta_1) = \alpha\Delta_1^{\alpha-1}(C - \Delta_1)^{1-\alpha} - (1-\alpha)\Delta_1^{\alpha}(C - \Delta_1)^{-\alpha}.$$

The first-order condition $f'(\Delta_1) = 0$ is equivalent to

$$\frac{\alpha}{\Delta_1} = \frac{1-\alpha}{C - \Delta_1} \iff \frac{\Delta_1}{\Delta_2} = \frac{\alpha}{1-\alpha}.$$

Hence $\Delta_1^* = \alpha C$, $\Delta_2^* = (1-\alpha)C$. Symmetry (equal utilities) requires $\Delta_1^* = \Delta_2^*$, which holds iff $\alpha = \frac{1}{2}$. Therefore, $\alpha \neq 1/2 \implies$ Not Symmetric which shows the contrapositive.

*($\Leftarrow$)* If $\alpha = \frac{1}{2}$, the objective $\Delta_1^{1/2}\Delta_2^{1/2} = (\Delta_1\Delta_2)^{1/2}$ is symmetric in players 1 and 2. On a symmetric feasible set with $d_1 = d_2$, if $x^*$ maximizes the objective then its swap also attains the same value. By strict quasi-concavity of the product on $\{\Delta_i > 0\}$, the maximizer is unique; hence $\Delta_1(x^*) = \Delta_2(x^*)$. Thus the symmetry axiom holds when $\alpha = \frac{1}{2}$.

$$\boxed{\text{The } \alpha\text{-solution satisfies symmetry iff } \alpha = \tfrac{1}{2}.}$$

**2) Threats–counterthreats representation and the relation between $p_1, p_2$.**

Define transformed utilities

$$V_1(x) := \Delta_1(x)^{\alpha}, \qquad V_2(x) := \Delta_2(x)^{1-\alpha}.$$

Then $x^*$ maximizes the Nash product $V_1(x)V_2(x)$ (with disagreement $V_i = 0$). Therefore, by the standard threats–counterthreats characterization of the Nash solution, for any $\tilde{x} \in X$ and any $p \in [0,1]$,

$$p\,V_i(\tilde{x}) + (1-p)\cdot 0 > V_i(x^*) \implies p\,V_{-i}(x^*) + (1-p)\cdot 0 > V_{-i}(\tilde{x}). \tag{†}$$

Translate (†) into the original $\Delta$-utilities.

<u>Premise (with $i = 1$):</u>

$$p\,V_1(\tilde{x}) > V_1(x^*) \iff p\,\Delta_1(\tilde{x})^{\alpha} > \Delta_1(x^*)^{\alpha} \iff p^{1/\alpha}\,\Delta_1(\tilde{x}) > \Delta_1(x^*).$$

Set $p_1 := p^{1/\alpha} \in [0,1]$. Then

$$p_1\, u_1(\tilde{x}) + (1-p_1)u_1(d) > u_1(x^*).$$

<u>Conclusion (corresponding $i=1$):</u>

$$p\, V_2(x^*) > V_2(\tilde{x}) \iff p\,\Delta_2(x^*)^{1-\alpha} > \Delta_2(\tilde{x})^{1-\alpha} \iff p^{1/(1-\alpha)}\,\Delta_2(x^*) > \Delta_2(\tilde{x}).$$

Set $p_2 := p^{1/(1-\alpha)} \in [0,1]$. Then

$$p_2\, u_2(x^*) + (1-p_2)u_2(d) > u_2(\tilde{x}).$$

Eliminating $p$ between $p_1 = p^{1/\alpha}$ and $p_2 = p^{1/(1-\alpha)}$ yields

$$p = p_1^{\alpha} \implies p_2 = (p_1^{\alpha})^{1/(1-\alpha)} = p_1^{\alpha/(1-\alpha)} \iff \boxed{p_1 = p_2^{\frac{1-\alpha}{\alpha}}}.$$

**3) Equivalence $p_1 = p_2 \iff \alpha = \frac{1}{2}$.**
From part 2,

$$p_1 = p_2^{\frac{1-\alpha}{\alpha}}.$$

If $\alpha = \frac{1}{2}$, the exponent equals 1 and hence $p_1 = p_2$. Conversely, if $0 < p_1 = p_2 < 1$, then

$$p_1 = p_1^{\frac{1-\alpha}{\alpha}} \Rightarrow \frac{1-\alpha}{\alpha} = 1 \Rightarrow \alpha = \tfrac{1}{2},$$

where we used that for $q \in (0,1)$, $q = q^k$ implies $k = 1$. Combining with part 1,

$$\boxed{\alpha = \tfrac{1}{2} \iff p_1 = p_2 \iff \text{the } \alpha\text{-solution is symmetric.}}$$