Supplementary Appendix for Commitment via Third-Party Contracts in Bilateral Trade: A Three-Way Equivalence

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S.1 Proof of Corollary 2.1

We prove four lemmata. Lemma S.1 derives an expression for expected royalty payments over the Pareto frontier, giving us the bound in point (i). Lemma S.2 shows the royalties, k^*_γ, are decreasing over $p \in \mu_\gamma([\underline{v}, \overline{v}])$, giving point (ii). Lemma S.3 demonstrates point (iii). Lemma S.4 shows that the interior of $[\underline{c}, \overline{c}] \cap \psi_{\mathcal{B},\gamma}([\underline{v}, \overline{v}])$ is non-empty if and only if $\mu_\gamma([\underline{v}, \overline{v}])$ has strictly positive Lebesgue measure. Therefore, whenever the interior of $[\underline{c}, \overline{c}] \cap \psi_{\mathcal{B},\gamma}([\underline{v}, \overline{v}])$ is non-empty, the buyer posts an interval of prices and from Lemma S.1 these are on average non-positive and from Lemma S.2 are decreasing in prices. Consequently, royalties are necessarily negative for high enough prices, giving point (iv).

Lemma S.1 *For $\gamma \in [0, 1]$, in the γ-maximal royalty scheme, k^*_γ, equilibrium expected royalty payments are non-positive and equal to*

$$K_\gamma = \left[\max\left\{0, \frac{1-2\gamma}{1-\gamma}\right\} - 1\right] \cdot \int_{\underline{v}}^{\overline{v}} G(\psi_{\mathcal{B},\gamma}(x))[1 - F(x)]dx \in [-W^e, 0]$$

Proof. Under the γ-maximal allocation, we mildly abuse notation and write the seller's surplus including their contract payments and their trade surplus as,

$$\pi_\gamma := \mathbb{E}_{\omega,v,c,x\sim Bern(a_{k^*_\gamma}(\omega,c,p_{k^*_\gamma}(v)))}[\pi(x, p_{k^*_\gamma}(v), k^*_\gamma; c, \omega)]$$
$$\hat{\pi}_\gamma := \mathbb{E}_{\omega,v,c,x\sim Bern(a_{k^*_\gamma}(\omega,c,p_{k^*_\gamma}(v)))}[\hat{\pi}(x, p_{k^*_\gamma}(v), k^*_\gamma; c, \omega)]$$

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Then,

$$
\begin{aligned}
\pi_\gamma &= \int_{\underline{v}}^{\overline{v}} \int_{\underline{c}}^{\overline{c}} \mathbb{1}_{\{\psi_{\mathcal{B},\gamma}(v) \geq c\}} (p_{k^*_\gamma}(v) - k^*_\gamma(p_{k^*_\gamma}(v)) - c) dG dF \\
&= \int_{\underline{v}}^{\overline{v}} \int_{\underline{c}}^{\overline{c}} \mathbb{1}_{\{\psi_{\mathcal{B},\gamma}(v) \geq c\}} (\psi_{\mathcal{B},\gamma}(v) - c) dG dF \qquad (\dagger)
\end{aligned}
$$

and,

$$
\begin{aligned}
\hat{\pi}_\gamma &= \int_{\underline{v}}^{\overline{v}} \int_{\underline{c}}^{\overline{c}} \mathbb{1}_{\{\psi_{\mathcal{B},\gamma}(v) \geq c\}} (p_{k^*_\gamma}(v) - c) dG dF \\
&= \int_{\underline{v}}^{\overline{v}} \int_{\underline{c}}^{\overline{c}} \mathbb{1}_{\{\psi_{\mathcal{B},\gamma}(v) \geq c\}} (\mu_\gamma(v) - c) dG dF \qquad (\ddagger)
\end{aligned}
$$

Denote the expected royalty payments in the $\gamma-$maximal outcome as K^*_γ. We have the following relation

$$
\hat{\pi}_\gamma - K^*_\gamma = \pi_\gamma
$$

Therefore, by ($\dagger$) and ($\ddagger$),

$$
\begin{aligned}
K^*_\gamma &= \hat{\pi}_\gamma - \pi_\gamma \\
&= \int_{\underline{v}}^{\overline{v}} \int_{\underline{c}}^{\overline{c}} \mathbb{1}_{\{\psi_{\mathcal{B},\gamma}(v) \geq c\}} (\mu_\gamma(v) - \psi_{\mathcal{B},\gamma}(v)) dG dF
\end{aligned}
$$

Now, using the expression for $\mu_\gamma(v)$ derived in the proof of Theorem 2, $\mu_\gamma(v) = v - \frac{1}{G(\psi_{\mathcal{B},\gamma}(v))} \int_{\underline{v}}^{v} G(\psi_{\mathcal{B},\gamma}(x)) dx$, we obtain by application of Fubini,

$$
\begin{aligned}
K^*_\gamma &= \int_{\underline{v}}^{\overline{v}} \int_{\underline{c}}^{\overline{c}} \mathbb{1}_{\{\psi_{\mathcal{B},\gamma}(v) \geq c\}} (\mu_\gamma(v) - \psi_{\mathcal{B},\gamma}(v)) dG dF \\
&= \int_{\underline{v}}^{\overline{v}} \int_{\underline{c}}^{\overline{c}} \mathbb{1}_{\{\psi_{\mathcal{B},\gamma}(v) \geq c\}} \begin{pmatrix} -\frac{1}{G(\psi_{\mathcal{B},\gamma}(v))} \int_{\underline{v}}^{v} G(\psi_{\mathcal{B},\gamma}(x)) dx \\ + \max\left\{0, \frac{1-2\gamma}{1-\gamma}\right\} \cdot \frac{1-F(v)}{f(v)} \end{pmatrix} dG dF \\
&= \int_{\underline{v}}^{\overline{v}} \int_{\underline{c}}^{\overline{c}} \mathbb{1}_{\{\psi_{\mathcal{B},\gamma}(v) \geq c\}} \left(-\frac{1-F(v)}{f(v)} + \max\left\{0, \frac{1-2\gamma}{1-\gamma}\right\} \cdot \frac{1-F(v)}{f(v)} \right) dG dF \\
&= \left[\max\left\{0, \frac{1-2\gamma}{1-\gamma}\right\} - 1 \right] \cdot \int_{\underline{v}}^{\overline{v}} G(\psi_{\mathcal{B},\gamma}(x)) [1 - F(x)] dx
\end{aligned}
$$

Clearly, K^*_γ is decreasing in γ. At $\gamma = 0$, $K^*_\gamma = 0$. At $\gamma = 1$, $\psi_{\mathcal{B},\gamma}(v) = v$ and

$\max\left\{0, \frac{1-2\gamma}{1-\gamma}\right\} - 1 = -1$. Notice that

$$\int_{\underline{v}}^{\overline{v}} \int_{\underline{c}}^{\overline{c}} \mathbb{1}\{v \geq c\}(\psi_{\mathcal{B},0}(v) - c) dG dF = W^e - \int_{\underline{v}}^{\overline{v}} G(x)[1 - F(x)]dx$$

So, by Lemma A.1,

$$K_1^* = -\int_{\underline{v}}^{\overline{v}} G(x)\big[1 - F(x)\big]\, dx = \int_{\underline{c}}^{\max\{\underline{c},\underline{v}\}} G(x)\, dx - W^e \geq -W^e.$$

As such, $K_\gamma^* \in [-W^e, 0]$. ☐

Lemma S.2 *If G is regular, k_γ^* is monotone decreasing over $p \in \mu_\gamma([\underline{v}, \overline{v}])$.*

Proof. Over $p \in \mu_\gamma(\psi_{\mathcal{B},\gamma}^{-1}([\underline{c}, \overline{c}])) \subseteq \mu_\gamma([\underline{v}, \overline{v}])$, royalties are implicitly defined by $k_\gamma^*(p_{k_\gamma^*}(v)) = p_{k_\gamma^*}(v) - \psi_{\mathcal{B},\gamma}(v)$. For ℓ a function of bounded variation, denote by $d\ell$ the Lebesgue-Stieltjes (signed) measure induced by ℓ, i.e., for $a < b$ $d\ell((a, b]) = \ell(b) - \ell(a)$. Note $d\ell$ is well-defined even if ℓ is non-differentiable. If $d\ell > (<)0$ then ℓ is increasing (decreasing). Because F has increasing hazard rate, $p_{k_\gamma^*}$, $\psi_{\mathcal{B},\gamma}$, and $\psi_{\mathcal{B},\gamma}^{-1}$ are monotone functions on compact intervals and are therefore bounded. Hence, the Lebesgue-Stieltjes measure is well-defined for these functions.

We first prove that $d(\psi_{\mathcal{B},\gamma}^{-1}) \leq \lambda$ where λ is the Lebesgue measure. Because F has increasing hazard rate, fix any $a < b$, then

$$\psi_{\mathcal{B},\gamma}(b) - \psi_{\mathcal{B},\gamma}(a) = b - a - \underbrace{\max\left\{0, \frac{2\gamma - 1}{1-\gamma}\right\}}_{\geq 0} \underbrace{\left(\frac{1 - F(b)}{f(b)} - \frac{1 - F(a)}{f(a)}\right)}_{\leq 0 \text{ as } F \text{ regular}} \geq b - a \tag{$\sim$}$$

For any $a < b$ in the range of $\psi_{\mathcal{B},\gamma}^{-1}$, because the inverse in increasing, there exists $x < y$ with $\psi_{\mathcal{B},\gamma}^{-1}(x) = a < b = \psi_{\mathcal{B},\gamma}^{-1}(y)$. Plugging into ($\sim$), $d(\psi_{\mathcal{B},\gamma}^{-1})((x, y]) = \psi_{\mathcal{B},\gamma}^{-1}(y) - \psi_{\mathcal{B},\gamma}^{-1}(x) \leq y - x$. Therefore, $\psi_{\mathcal{B},\gamma}^{-1}$ has Lipschitz constant 1 and we have $\frac{d(\psi_{\mathcal{B},\gamma}^{-1})}{d\lambda} \leq 1$. As a result, for any non-negative integrable function ℓ, $\int \ell d(\psi_{\mathcal{B},\gamma}^{-1}) \leq \int \ell d\lambda$.

Because $k_\gamma^*(p_{k_\gamma^*}(v)) = p_{k_\gamma^*}(v) - \psi_{\mathcal{B},\gamma}(v)$, we have the measure identity, $d(k_\gamma^* \circ p_{k_\gamma^*}) = dp_{k_\gamma^*} - d\psi_{\mathcal{B},\gamma}$. Fix $v_1, v_2 \in [\underline{v}, \overline{v}]$ with $v_1 < v_2$ offering prices $p_{k_\gamma^*}(v_1), p_{k_\gamma^*}(v_1) \in$

$\mu_\gamma([\underline{v},\overline{v}])$. We have, $\underline{c} < \psi_{\mathcal{B},\gamma}(v_1) \le \psi_{\mathcal{B},\gamma}(v_1)$. Then,

$$
\begin{aligned}
d(k^*_\gamma \circ p_{k^*_\gamma})((v_1,v_2]) &= dp_{k^*_\gamma}((v_1,v_2]) - d\psi_{\mathcal{B},\gamma}((v_1,v_2]) \\
&= \left[\begin{array}{c} v_2 - v_1 - d\left[\frac{1}{G(\psi_{\mathcal{B},\gamma}(\cdot))}\int_{\underline{v}}^{\cdot} G(\psi_{\mathcal{B},\gamma}(x))dx\right]((v_1,v_2]) \\ -d\psi_{\mathcal{B},\gamma}((v_1,v_2]) \end{array} \right] \\
&= \int_{v_1}^{v_2} \left[\underbrace{\frac{g(\psi_{\mathcal{B},\gamma}(v))}{G(\psi_{\mathcal{B},\gamma}(v))^2}\int_{\underline{v}}^{v} G(\psi_{\mathcal{B},\gamma}(x))dx}_{:=B(v)} - 1 \right] d\psi_{\mathcal{B},\gamma}(v)
\end{aligned}
$$

Because $\psi_{\mathcal{B},\gamma}$ and G/g are both increasing,

$$
\begin{aligned}
B(v) &= \frac{g(\psi_{\mathcal{B},\gamma}(v))}{G(\psi_{\mathcal{B},\gamma}(v))^2}\int_{\underline{v}}^{v} G(\psi_{\mathcal{B},\gamma}(x))dx \\
&= \frac{g(\psi_{\mathcal{B},\gamma}(v))}{G(\psi_{\mathcal{B},\gamma}(v))^2}\int_{\underline{v}}^{v} \frac{G(\psi_{\mathcal{B},\gamma}(x))}{g(\psi_{\mathcal{B},\gamma}(x))} g(\psi_{\mathcal{B},\gamma}(x))dx \\
&< \frac{1}{G(\psi_{\mathcal{B},\gamma}(v))}\int_{\underline{v}}^{v} g(\psi_{\mathcal{B},\gamma}(x))dx \\
&= \frac{1}{G(\psi_{\mathcal{B},\gamma}(v))}\int_{\psi_{\mathcal{B},\gamma}(\underline{v})}^{\psi_{\mathcal{B},\gamma}(v)} g(u)d(\psi^{-1}_{\mathcal{B},\gamma}(u)) \\
&\le \frac{1}{G(\psi_{\mathcal{B},\gamma}(v))}\int_{\psi_{\mathcal{B},\gamma}(\underline{v})}^{\psi_{\mathcal{B},\gamma}(v)} g(u)du \\
&= \frac{G(\psi_{\mathcal{B},\gamma}(v)) - G(\psi_{\mathcal{B},\gamma}(\underline{v}))}{G(\psi_{\mathcal{B},\gamma}(v))}
\end{aligned}
$$

As a result, $B(v) < 1$. As $d\psi_{\mathcal{B},\gamma} > 0$,

$$
d(k^*_\gamma \circ p_{k^*_\gamma})((v_1,v_2]) = \int_{v_1}^{v_2} (B(v) - 1)d\psi_{\mathcal{B},\gamma}(v) < 0
$$

Because $dp_{k^*_\gamma} > 0$, this implies $dk^*_\gamma < 0$ and hence that k^*_γ is strictly decreasing over $p \in \mu_\gamma([\underline{v},\overline{v}])$. □

Lemma S.3 *For $1/2 > \gamma > \gamma'$ and $p \in \mu_\gamma([\underline{v},\overline{v}]) \cap \mu_{\gamma'}([\underline{v},\overline{v}])$, $k^*_\gamma(p) < k^*_{\gamma'}(p)$.*

Proof. As in Lemma S.2, we note royalties are implicitly defined by $k^*_\gamma(p_{k^*_\gamma}(v)) =$

$p_{k^*_\gamma}(v) - \psi_{\mathcal{B},\gamma}(v)$. Because $\psi_{\mathcal{B},\gamma}(v)$ for fixed v is strictly increasing in γ,

$$p_{k^*_\gamma}(v) = \mu_\gamma(v) = \begin{cases} \mathbb{E}_c[\psi^{-1}_{\mathcal{B},\gamma}(c) \mid c \leq \psi_{\mathcal{B},\gamma}(v)] & \text{if } \gamma \leq 1/2, \\ \mathbb{E}_c[c \mid c \leq v] & \text{if } \gamma > 1/2 \end{cases}$$

is increasing γ for $\gamma < 1/2$. Therefore, since $\frac{d}{d\gamma}(k^*_\gamma \circ p_{k^*_\gamma})(v) = \frac{dk^*_\gamma(p_{k^*_\gamma}(v))}{d\gamma}\frac{dp_{k^*_\gamma}(v)}{d\gamma}$ if $\frac{d}{d\gamma}k^*_\gamma(p_{k^*_\gamma}(v)) < 0$ then $\frac{dk^*_\gamma(p_{k^*_\gamma}(v))}{d\gamma} < 0$. Note that any value type v posting $p \in \mu_\gamma([\underline{v}, \overline{v}]) \cap \mu_{\gamma'}([\underline{v}, \overline{v}])$ must satisfy $\psi_{\mathcal{B},\tilde{\gamma}}(v) > \underline{c}$ for all $\tilde{\gamma} \in [\gamma', \gamma]$. Consider for $\gamma < 1/2$,

$$\begin{aligned}
\frac{d}{d\gamma}k^*_\gamma(p_{k^*_\gamma}(v)) &= \frac{d\mu_\gamma(v)}{d\gamma} - \frac{d\psi_{\mathcal{B},\gamma}(v)}{d\gamma} \\
&= \frac{d}{d\gamma}\left(v - \frac{1}{G(\psi_{\mathcal{B},\gamma}(v))}\int_{\underline{v}}^{v} G(\psi_{\mathcal{B},\gamma}(x))dx\right) - \frac{1}{(1-\gamma)^2}\frac{1-F(v)}{f(v)} \\
&= \left(\begin{array}{c} \frac{1}{(1-\gamma)^2}\frac{1-F(v)}{f(v)}\left[\frac{g(\psi_{\mathcal{B},\gamma}(v))}{G(\psi_{\mathcal{B},\gamma}(v))^2}\int_{\underline{v}}^{v} G(\psi_{\mathcal{B},\gamma}(x))dx\right] \\ -\frac{1}{(1-\gamma)^2}\frac{1}{G(\psi_{\mathcal{B},\gamma}(v))}\int_{\underline{v}}^{v} g(\psi_{\mathcal{B},\gamma}(x))\frac{1-F(x)}{f(x)}dx - \frac{1}{(1-\gamma)^2}\frac{1-F(v)}{f(v)} \end{array}\right) \\
&\leq \frac{1}{(1-\gamma)^2}\frac{1-F(v)}{f(v)}\left[\begin{array}{c} \frac{g(\psi_{\mathcal{B},\gamma}(v))}{G(\psi_{\mathcal{B},\gamma}(v))^2}\int_{\underline{v}}^{v} G(\psi_{\mathcal{B},\gamma}(x))dx \\ -\frac{1}{G(\psi_{\mathcal{B},\gamma}(v))}\int_{\underline{v}}^{v} g(\psi_{\mathcal{B},\gamma}(x))dx - 1 \end{array}\right] \\
&= \frac{1}{(1-\gamma)^2}\frac{1-F(v)}{f(v)}\left[\begin{array}{c} \frac{g(\psi_{\mathcal{B},\gamma}(v))}{G(\psi_{\mathcal{B},\gamma}(v))^2}\int_{\underline{v}}^{v} G(\psi_{\mathcal{B},\gamma}(x))dx \\ -\frac{1}{G(\psi_{\mathcal{B},\gamma}(v))}\int_{\underline{v}}^{v}\frac{g(\psi_{\mathcal{B},\gamma}(x))}{G(\psi_{\mathcal{B},\gamma}(x))}G(\psi_{\mathcal{B},\gamma}(x))dx - 1 \end{array}\right] \\
&\leq \frac{1}{(1-\gamma)^2}\frac{1-F(v)}{f(v)}\left[\begin{array}{c} \frac{g(\psi_{\mathcal{B},\gamma}(v))}{G(\psi_{\mathcal{B},\gamma}(v))^2}\int_{\underline{v}}^{v} G(\psi_{\mathcal{B},\gamma}(x))dx \\ -\frac{g(\psi_{\mathcal{B},\gamma}(v))}{G(\psi_{\mathcal{B},\gamma}(v))^2}\int_{\underline{v}}^{v} G(\psi_{\mathcal{B},\gamma}(x))dx - 1 \end{array}\right] \\
&= -\frac{1}{(1-\gamma)^2}\frac{1-F(v)}{f(v)} < 0
\end{aligned}$$

Therefore, for $1/2 > \gamma > \gamma'$ and $p \in \mu_\gamma([\underline{v}, \overline{v}]) \cap \mu_{\gamma'}([\underline{v}, \overline{v}])$, $k^*_\gamma(p) < k^*_{\gamma'}(p)$. □

Lemma S.4 *The interior of* $[\underline{c}, \overline{c}] \cap \psi_{\mathcal{B},\gamma}([\underline{v}, \overline{v}])$ *is non-empty if and only if* $\mu_\gamma([\underline{v}, \overline{v}])$ *has strictly positive Lebesgue measure.*

Proof. Recall $\mu_\gamma(v) = \mathbb{E}_c[\psi^{-1}_{\mathcal{B},\gamma}(c) \mid c \le \psi_{\mathcal{B},\gamma}(v)]$. Then,

$$\begin{aligned}
\mu_\gamma(\underline{v}) &= \underline{v} \\
\mu_\gamma(\overline{v}) &= \mathbb{E}_c[\psi^{-1}_{\mathcal{B},\gamma}(c) \mid c \le \psi_{\mathcal{B},\gamma}(\overline{v})] \\
&= \overline{v} - \frac{1}{G(\psi_{\mathcal{B},\gamma}(\overline{v}))}\int_{\underline{v}}^{\overline{v}} G(\psi_{\mathcal{B},\gamma}(x))dx
\end{aligned}$$

We first prove that if $[\underline{c},\overline{c}] \cap \psi_{\mathcal{B},\gamma}([\underline{v},\overline{v}])$ has non-empty interior, $\mu_\gamma([\underline{v},\overline{v}])$ has strictly positive Lebesgue measure. If the interior is non-empty then either $\overline{v} > \psi^{-1}_{\mathcal{B},\gamma}(\underline{c}) > \underline{v}$ or $\overline{v} > \psi^{-1}_{\mathcal{B},\gamma}(\overline{c}) > \underline{v}$.

If $\overline{v} > \psi^{-1}_{\mathcal{B},\gamma}(\overline{c}) > \underline{v}$, $G(\psi_{\mathcal{B},\gamma}(\overline{v})) = 1$ but $\int_{\underline{v}}^{\overline{v}} G(\psi_{\mathcal{B},\gamma}(x))dx < \overline{v} - \underline{v}$, so $\mu_\gamma(\overline{v}) > \underline{v} = \mu_\gamma(\underline{v})$. Because μ_γ is continuous, by intermediate value theorem $\mu_\gamma([\underline{v},\overline{v}]) = [\mu_\gamma(\underline{v}), \mu_\gamma(\overline{v})]$ and $\mu_\gamma(\overline{v}) - \mu_\gamma(\underline{v}) > 0$.

If $\overline{v} > \psi^{-1}_{\mathcal{B},\gamma}(\underline{c}) > \underline{v}$, $\frac{1}{G(\psi_{\mathcal{B},\gamma}(\overline{v}))} \ge 1$ and $\int_{\underline{v}}^{\overline{v}} G(\psi_{\mathcal{B},\gamma}(x))dx < \overline{v} - \underline{v}$, and similar logic applies as in the other case.

Second, that if $\mu_\gamma([\underline{v},\overline{v}])$ has strictly positive Lebesgue measure then the interior of $[\underline{c},\overline{c}] \cap \psi_{\mathcal{B},\gamma}([\underline{v},\overline{v}])$ is non-empty follows immediately from the definition of μ_γ. □

S.2 Computation of UPS

We have $c \sim U[0,1]$, $v \sim U[0,1]$. First, in any buyer incentive compatible mechanism, by payoff equivalence and Fubini, the buyer and seller surpluses are equal to

$$\begin{aligned}
BS(q) &= \int_0^1 \int_0^1 q(c,v)[1-v]dvdc \\
SS(q) &= \int_0^1 \int_0^1 q(c,v)[2v-1-c]dvdc
\end{aligned}$$

We derive the contract-implementable frontier. From ??, the frontier is spanned by $q^*_\gamma(c,v) = \mathbb{1}\{\psi_{\mathcal{B},\gamma}(v) \ge c\}$. Define $\Gamma = \max\{0, (1-2\gamma)/(1-\gamma)\}$. Then, the contract-implementable utility possibility frontier is defined by

$$\partial\mathbb{U}_C := \left\{(U_B, U_S) = \left(\frac{1}{6(1+\Gamma)^2}, \frac{\Gamma}{3(1+\Gamma)^2}\right) \,\middle|\, \gamma \in [0,1]\right\}$$

This gives the blue boundary for $U_B \geq 1/24$.

For the boundary when $U_B \in [0, 1/24)$, we solve the problem

$$V(\overline{u}) = \max_q SS(q)$$

$$st. \begin{cases} \int_0^1 \int_0^1 q(c,v)[2v-1-c]dvdc \geq 0 \\ \overline{u} = \int_0^1 \int_0^1 q(c,v)[1-v]dvdc \end{cases}$$

for $\overline{u} \in [0, 1/24)$. This solves to $V(\overline{u}) = \frac{\sqrt{6\overline{u}}}{3} - 2\overline{u}$. This defines the contract feasible set as the origin, $(U_B, U_S) = (0, 0)$ is feasible and the utility possibility set is convex.

For the Myerson-Satterthwaite frontier, we employ the techniques of Ledyard and Palfrey (1999) and Williams (1987). Using Theorem 1 of Myerson and Satterthwaite (1983), we solve the program

$$\max_q \gamma BS(q) + (1-\gamma)SS(q)$$

$$st. \int_0^1 \int_0^1 q(c,v)\,[2v-1-2c]\,dvdc \geq 0$$ [1]

Consider the Lagrangian,

$$\begin{aligned} \mathcal{L} &= \gamma BS(q) + (1-\gamma)SS(q) + \lambda \int_0^1 \int_0^1 q(c,v)(2v-1-2c)dcdv \\ &= \int_0^1 \int_0^1 q(c,v)\phi_{\gamma,\lambda}(c,v)dcdv \end{aligned}$$

where $\phi_{\gamma,\lambda}(c,v) := \gamma(1-v) + (1-\gamma)(2v-1-c) + \lambda(2v-1-2c)$. The optimal allocation is then $q_{\gamma,\lambda}(c,v) = \mathbb{1}\{\phi_{\gamma,\lambda}(c,v) \geq 0\}$. The problem becomes selecting $\lambda_\gamma = \lambda$ such that the ex ante budget constraint is satisfied and, if binding, satisfied with equality. One may verify that if $\lambda_\gamma = 0$, $q_{\gamma,\lambda_\gamma}$ violates the budget constraint for all $\gamma > 0$. At $\gamma = 0$, the unconstrained maximum is achieved by $q_{0,0}(c,v) = \mathbb{1}\{v \geq (1+c)/2\}$.

[1] Strictly speaking, this is the relaxed problem absent the monotonicity constraints implied by incentive compatibility. Nonetheless, because cost and value distributions are regular (uniform), the solution to this relaxed problem naturally satisfies these additional constraints.

For $\gamma > 0$, $\lambda_\gamma > 0$ is chosen to satisfy

$$\int_0^1 \int_0^1 q_{\gamma,\lambda_\gamma}(c,v)[2v-1-2c]dcdv = 0$$

Given the formula for $q_{\gamma,\lambda_\gamma}$, the equation reduces to a quadratic in λ_γ. Selecting the postiive root, we find $\lambda_\gamma = \frac{2\gamma-1+\sqrt{3\gamma^2-3\gamma+1}}{2}$ note that $\lambda_0 = 0$ so encompasses the edge case. We may then derive buyer and seller surpluses over the frontier in closed form. Then,

$$BS(q^{MS}_{\gamma,\lambda_\gamma}) = \frac{(\sqrt{3\gamma^2-3\gamma+1}+1)^3}{48(\gamma+\sqrt{3\gamma^2-3\gamma+1})(1-\gamma+\sqrt{3\gamma^2-3\gamma+1})^2}$$

$$SS(q^{MS}_{\gamma,\lambda_\gamma}) = \frac{(\sqrt{3\gamma^2-3\gamma+1}+1)^3}{48(\gamma+\sqrt{3\gamma^2-3\gamma+1})^2(1-\gamma+\sqrt{3\gamma^2-3\gamma+1})}$$

so the MS-ex ante frontier is

$$\partial\mathbb{U}_{MS} := \{(U_B, U_S) = (BS(q^{MS}_{\gamma,\lambda_\gamma}), SS(q^{MS}_{\gamma,\lambda_\gamma})) \mid \gamma \in [0,1]\}$$

To get the left-hand side of the MS 'teardrop', we solve the program

$$W(\overline{u}) = \max_q SS(q)$$

$$st. \begin{cases} \int_0^1 \int_0^1 q(c,v)[2v-1-2c]dvdc \geq 0 \\ \overline{u} = \int_0^1 \int_0^1 q(c,v)[1-v]dvdc \end{cases}$$

For $\overline{u} \in [0, 1/24)$ This has solution $W(\overline{u}) = V(\overline{u}) = \frac{\sqrt{6\overline{u}}}{3} - 2\overline{u}$.

In Myerson-Satterthwaite, buyer and seller are symmetric in the mechanism. Therefore, since cost and value distributions are also symmetric in the uniform case we are considering, we can trace out the right-hand side of the MS 'teardrop' by mapping out the graph $(W(\underline{u}), \underline{u})$ for $\underline{u} \in [0, 1/24)$, i.e., by reflecting the graph about the line $U_B = U_S$.