Multidimensional Monotonicty and Economic Applications

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LSE Reading Group

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Motivation

Central object in mechanism design is allocation rule. Often restricted by monotonicity.

- With one (or symmetric) agent(s), usually require interim allocation rule $q : [0, 1] \to [0, 1]$ monotone increasing.
- With n agents, generalise to *cyclic monotonicty* of interim rules $(q_1, q_2, \ldots, q_n) : [0, 1]^n \to [0, 1]^n$.

With convex/linear objective, extreme points are maximisers (Bauer).

- In one-agent case, extreme points are indicator functions $f_y(x) = \mathbb{1}_{\{x \geq y\}}$
- Extreme points of cyclic monotone functions intractible, even in $n = 2$ case.

Can we progress by keeping things as univariate as possible?

The Paper: A Roadmap

Start with monotone functions $f : [0,1]^n \to [0,1]$
Seek to characterise extreme points.

- **Prop 1**: Represent any multivariate monotone function as a *univariate* probability distribution over sets.
- **Thm 1/Prop 2**: Characterise extreme points in space of multivariate monotone functions subject to linear constraints.

In applications, want monotonicity of marginals $q_i(x_i) = \int_{[0,1]^{n-1}} f(x) dx_{-i}$. What are the extreme points of $(q_1, \ldots, q_n)$ which are rationalised as marginals of some f (not necessarily monotone)?

- **Lma 2/3**: Monotone marginals can be rationalised by monotone f. Extreme points of set of marginals is subset of extreme points of f.
- **Thm 2/3/4**: Characterise extreme points in space of monotone marginals based on extreme points of $f : [0,1]^n \to [0,1]$.

Prop 8/9: Apply results. Focus on interim efficient frontier in bilateral trade.

Definitions

Let $\mathcal{F} \subset L^1([0,1]^n)$ be all integrable monotone functions $f : [0,1]^n \to [0,1]$ (convex + compact).

Up-Set

A set $A \subset [0,1]^n$ is an **Up-Set** if $x \in A, y \geq x \implies y \in A$

Extreme Point

For a convex set C, $x \in C$ is an **Extreme Point** if it is **not** a linear combination of two distinct elements. Denote set of extreme points as $\mathrm{Ext}(C)$.

Nested Up-Sets

A family of up-sets $\{A_i\}_{i \in \mathcal{I}}$ is **Nested** if $A_i \subset A_{i'}$ for any $i < i'$.

To Choquet...

Lemma 1 (Choquet, 1954)

An $f \in \mathcal{F}$ is an extreme point if and only if $f = \mathbb{1}_A$ for some up-set $A \subset [0,1]^n$.

Why do we care about extreme points?

Choquet Representation

For any $f \in \mathcal{F}$, there exists probability measure $\mu \in \Delta([0,1])$ and collection of up-sets $\{A_r\}$ such that

$$f = \int_{[0,1]} \mathbb{1}_{A_r} d\mu(r)$$

Given objective is linear or convex functional, $\mathbf{L} : \mathcal{F} \to \mathbb{R}$, for any $f \in \mathcal{F}$, by Jensen's

$$\mathbf{L}(f) = \mathbf{L}\left(\int_{[0,1]} \mathbb{1}_{A_r} d\mu(r)\right) \leq \int_{\mathrm{Ext}(\mathcal{F})} \mathbf{L}(\mathbb{1}_{A_r}) d\mu(r)$$

To Choquet... And Beyond

Choquet representation not tractible when including other constraints.

- Up-sets may be non-nested.
- Complexity blows up when mixing extreme points in Choquet representation.

Nesting Representation

$f \in \mathcal{F}$ if and only if there exist a **Unique** collection of **Nested** up-sets $\{A_r\}$ and a **Unique** probability measure $\mu \in \Delta([0,1])$ such that

$$f = \int_{[0,1]} \mathbb{1}_{A_r} d\mu(r)$$

Proof Idea: "if" part immediate as monotonicity preserved by mixtures. For "only if", write $f \in \mathcal{F}$ as

$$f(x) = \int_{[0,1]} \mathbb{1}_{\{f(x) \geq 1-r\}} dr, \qquad A_r := \{x \in [0,1]^n \mid f(x) \geq 1-r\}$$

Adding Constraints

Given finite collection $\{\phi^j\}_{j=1}^m \subset L^\infty([0,1]^n)$ and $\{\eta^j\}_{j=1}^m \subset \mathbb{R}$, what are extreme points of

$$\overline{\mathcal{F}} := \left\{ f \in \mathcal{F} \;\middle|\; \int_{[0,1]^n} f(x)\phi^j(x)dx \leq \eta^j, \forall j \in [m] \right\}$$

Without nesting rep, extreme points could be mixture of 2^{m+1} indicator functions of possibly non-nested upsets.

Theorem 1

Any $f \in \mathrm{Ext}(\overline{\mathcal{F}})$ is a mixture of **at most** $m+1$ indicator functions $\{\mathbb{1}_{A_j}\}_{j=1}^{m+1}$ where $\{A_j\}_{j=1}^{m+1}$ where each A_j is an up-set. (**Prop 2**, partial converse)

Reduces complexity $2^{m+1} \mapsto m+1 \implies$ exponential reduction. Graphics

Rationalisable Monotone Functions

Rationalisable Marginals

A tuple $q = (q_1, \dots, q_n)$ of non-decreasing, left-continuous mappings $[0,1] \mapsto [0,1]$ is **rationalisable**, if there exists $f : [0,1]^n \to [0,1]$ (perhaps not monotone) such that for all i, x_i

$$q_i(x_i) = \int_{[0,1]^{n-1}} f(x) dx_{-i}$$

Let $\mathcal{Q}$ be collection of rationalisable q's.

Want to relate monotone marginals to earlier work on monotone $f \in \mathcal{F}$.

Lemma 2

A tuple of one-dimensional monotone functions $q = (q_1, q_2, \dots, q_n)$ is rationalisable if and only if it can be rationalised by an $f \in \mathcal{F}$. That is, $\mathcal{Q} = P(\mathcal{F})$ for some linear projection operator P.

Extreme Points of Rationalisable Marginals

Lemma 3: Affine Mapping Lemma

Given $\mathcal{Q} = P(\mathcal{F})$ for some linear projection P,

$$\text{Ext}(\mathcal{Q}) \subseteq P(\text{Ext}(\mathcal{F}))$$

Theorem 2: Part (i)

Every extreme point $q \in \text{Ext}(\mathcal{Q})$ is rationalised by $f = \mathbb{1}_A$ for some up-set $A \subseteq [0,1]^n$.

Proof Idea: Explicitly, define projection map $P : \mathcal{F} \to \mathcal{Q}$ as

$$P(f) = \left(\int_{[0,1]^{n-1}} f(x) dx_{-1}, \ldots, \int_{[0,1]^{n-1}} f(x) dx_{-n} \right)$$

By Lemma 3, $\text{Ext}(\mathcal{Q}) \subseteq P(\text{Ext}(\mathcal{F}))$. Conclude from characterisation of $\text{Ext}(\mathcal{F})$.

Rationalisable Marginals with Constraints

Fix family $\{\phi_i^j\}_{1\leq i\leq n}^{1\leq j\leq m} \subset L^\infty([0,1])$ and $\{\eta^j\}_{j=1}^m \subset \mathbb{R}$. What are extreme points of

$$\overline{\mathcal{Q}} := \left\{ q \in \mathcal{Q} \;\middle|\; \sum_{i=1}^n \int_{[0,1]} q_i(x_i)\phi_i^j(x_i)dx_i \leq \eta^j, \forall j \in [m] \right\}$$

Theorem 2: Part (ii)

Every extreme point $q \in \mathrm{Ext}(\overline{\mathcal{Q}})$ is rationalised by a mixture of $\{\mathbb{1}_{A_j}\}_{j=1}^{m+1}$ for some nested up-sets $\{A_j\}_{j=1}^{m+1}$

Proof Idea: Similarly, $\overline{\mathcal{Q}} = P(\overline{\mathcal{F}})$. Use characterisation of extreme points of $\overline{\mathcal{F}}$. Necessary conditions for extreme points of $\mathcal{Q}$ and $\overline{\mathcal{Q}}$. In general, not sufficient. But can be sharpened when $n = 2$.

Rationalisable Monotone Pairs

Restrict attention to $n = 2$ (2 agents).

Theorem 3

$q = (q_1, q_2)$ is in $\mathrm{Ext}(\mathcal{Q})$ if and only if q is rationalised by $\mathbb{1}_A$ for some up-set $A \subseteq [0,1]^2$. Moreover, every extreme point of $\mathcal{Q}$ is uniquely rationalised.

Proof Idea: Necessity immediate from Theorem 2. WTS any $q \in \mathcal{Q}$ rationalised by $\mathbb{1}_A$ for up-set $A \subseteq [0,1]^2$ is an extreme point.

- Instead show q is **exposed point**. That is, there is a cts. linear functional whose unique maximiser is q.
- As A up-set $A = \{(x_1, x_2) : x_2 \geq g(x_1)\}$. Let $\phi_1(x_1) := -g(x_1), \phi_2(x_2) := x_2$.
- Notice, $\phi_1(x_1) + \phi(x_2) = x_2 - g(x_2)$. Greater than 0 if and only if $(x_1, x_2) \in A$.
- So $f = \mathbb{1}_A$ is unique maximiser of

$$L(f) := \int_{[0,1]^2} (\phi_1(x_1) + \phi_2(x_2)) f(x_1, x_2) dx \implies (q_1, q_2) \in \mathrm{Ext}(\mathcal{Q})$$

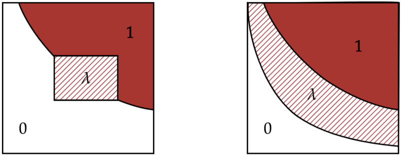
Monotone Pairs with One Constraint

Add one constraint to 2-agent characterisation. Fix $\psi_1, \psi_2 \in L^\infty([0,1])$, and $\eta \in \mathbb{R}$. Recall

$$\overline{\mathcal{Q}} = \left\{ q \in \mathcal{Q} \;\middle|\; \int_{[0,1]} q_1(x_1)\psi_1(x_1)dx_1 + \int_{[0,1]} q_2(x_2)\psi_2(x_2)dx_2 \leq \eta \right\}$$

Theorem 4

Any $q \in \mathrm{Ext}(\overline{\mathcal{Q}})$ is rationalised by a mixture of $\mathbb{1}_A$ and $\mathbb{1}_{A'}$ where $A' \subseteq A \subseteq [0,1]^2$ are nested up-sets that differ by at most a rectangle. Moreover, this is the unique rationalising function of q.



(a) An Extreme Point

(b) Not an Extreme Point

Projections with Other Measures

Fix two CDFs, G_1, G_2, cts. and full support on $[\underline{x}_1, \overline{x}_1], [\underline{x}_2, \overline{x}_2]$, respectively.

(G_1, G_2)-Rationalisability

$q = (q_1, q_2)$ are (G_1, G_2)-**rationalised** by $f : [\underline{x}_1, \overline{x}_1] \times [\underline{x}_2, \overline{x}_2] \to [0, 1]$ if

$$(q_1(x_1), q_2(x_2)) = \left(\int_{[\underline{x}_2, \overline{x}_2]} f(x_1, x_2) dG_2(x_2), \int_{[\underline{x}_1, \overline{x}_1]} f(x_1, x_2) dG_1(x_1) \right)$$

Let $\mathcal{Q}^{(G_1, G_2)}$ be set of (q_1, q_2) non-decreasing functions that are (G_1, G_2) rationalisable. Define $\overline{\mathcal{Q}}^{(G_1, G_2)}$ analogously as before.

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Corollary 2

Any $q \in \mathrm{Ext}(\overline{\mathcal{Q}}^{(G_1, G_2)})$ is rationalised by a mixture of $\mathbb{1}_A$ and $\mathbb{1}_{A'}$ where $A \subseteq A' \subseteq [\underline{x}_1, \overline{x}_1] \times [\underline{x}_2, \overline{x}_2]$ are nested up-set that differ by at most a rectangle. Moreover, this is the unique rationalising function of q.

Bilateral Trade: Interim Frontier

Suppose a buyer and seller bargaining over sale of single indivisible good.

1. Private-value buyer, $v \sim F$. Density f with full support over $[\underline{v}, \overline{v}]$.
2. Private-cost seller, $c \sim G$. Density g with full support over $[\underline{c}, \overline{c}]$.
3. Trade in **direct & incentive compatible** mechanism

$$(p, t) : [\underline{v}, \overline{v}] \times [\underline{c}, \overline{c}] \to [0, 1] \times \mathbb{R}$$

4. Let $U_B(v)$ and $U_S(c)$ be interim payoffs.

Want interim constrained Pareto frontier. Let $\Lambda_B \in \Delta([\underline{v}, \overline{v}]), \Lambda_S \in \Delta([\underline{c}, \overline{c}])$ be welfare weights. Solve

$$\max_{(p,t)} \int_{\underline{v}}^{\overline{v}} U_B(v) d\Lambda_B(v) + \int_{\underline{c}}^{\overline{c}} U_S(c) d\Lambda_S(c)$$

subject to IC + IR for B and S.

Bilateral Trade: Interim Frontier

Definition: Markup-Pooling Mechanism

A mechanism (p, t) is **markup-pooling** if there exists non-decreasing ϕ, an interval $I = [c_L, c_H]$, and constant $k \in [0, 1]$ such that

1. If $c \notin I$, B and S trade if and only if $v \geq \phi(c)$
2. If $c \in I$, let $\hat{c} = c_H$ with probability k and $\hat{c} = c_L$ with $1 - k$ probability. Then, trade occurs if and only $v \geq \phi(\hat{c})$.

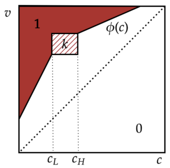


Figure 6: A Markup-Pooling Mechanism

Bilateral Trade: Interim Frontier

Proposition 8

For any welfare weights Λ_B, Λ_S, there exists a markup-pooling mechanism maximising expected welfare.

Proof: Let $\mathcal{M}$ be collection of IC + IR mechanisms. Define possibility set

$$\mathcal{U} := \{(U_B^M, U_S^M) : M \in \mathcal{M}\}$$

WTS extreme points of $\mathcal{U}$ are markup-pooling $\implies$ claim by Bauer

- Hard problem, instead show that for any $z \in \mathbb{R}_+$, extreme points of

$$\mathcal{U}_z := \{(U_B^M, U_S^M) : M \in \mathcal{M}, U_B^M(\underline{v}) = z\}$$

  are markup-pooling.
- Benefit of $\mathcal{U}_z$ — can fix one agent's payoff, gets back to $n = 1$ case.

Bilateral Trade: Interim Frontier

By Myerson Sattherthwaite (1983),

$$\Pi(P_S, P_B) := \int_{\underline{v}}^{\overline{v}} P_B(v) \left(v - \frac{1 - F(v)}{f(v)} \right) dF - \int_{\underline{c}}^{\overline{c}} P_S(c) \left(c + \frac{G(c)}{g(c)} \right) dG$$
$$= U_B(\underline{v}) + U_S(\overline{c})$$

where P_B, P_S are interim alloc rules.

- Define

$$\mathcal{Q} = \{(P_B, P_S) : (F, G)\text{−rationalisable, } P_B \text{ non-decreasing}, P_S \text{ non-increasing}\}$$

- Let

$$\overline{\mathcal{Q}}_z = \{(P_B, P_S) \in \mathcal{Q} : \Pi(P_B, P_S) \geq z\}$$

Bilateral Trade: Interim Frontier

Now, define linear operator $\mathbf{L}_z : \overline{\mathcal{Q}}_z \to \mathcal{U}_z$ as

$$\mathbf{L}[P_B, P_S](v, c) = \left(\int_{\underline{v}}^{v} P_B(x)dx + z,\ \int_{c}^{\overline{c}} P_S(x)dx + \Pi(P_B, P_S) - z \right)$$

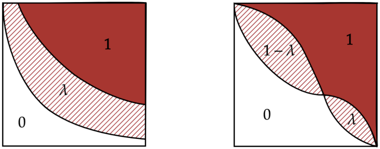
Then, $\mathcal{U}_z = \mathbf{L}(\overline{Q}_z)$.

- Therefore, $\text{Ext}(\mathcal{U}_z) \subseteq \text{Ext}(\overline{Q}_z)$ (Lma 3).
- Any point of $\text{Ext}(\overline{Q}_z)$ must have form

$$\mathbb{1}_{A_1} + \lambda \mathbb{1}_{A_2 \setminus A_1}$$

  where $A_1 \subseteq A_2$ are nested up-sets in $(v, -c)$ space and $A_2 \setminus A_1$ is a rectangle (Thm 3).
- Any such mechanism is implementable by markup-pooling.

Extreme Points of $\overline{\mathcal{F}}$



(a) An Extreme Point (b) Not an Extreme Point

Figure 2: Extreme Points of $\overline{\mathcal{F}}$

Figure 1: Extreme points of $\overline{\mathcal{F}}$

Both (a) and (b) represent function $f(x) = \lambda \mathbb{1}_A + (1-\lambda)\mathbb{1}_{A'}$. In (a), A, A' are nested and f is an extreme point. In (b), A, A' are not nested, so f is not an extreme point. Go Back